Distributed Private Heavy Hitters

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A motivating problem: Website referrals

A popular website wants to know who the top referrer is.
A motivating problem: Website referrals

A popular website wants to know who the top referrer is.

Each user knows where he arrived from, but he doesn’t want to make this information public (may be embarrassing)
How to protect privacy?

**Differential Privacy**

- Rigorous, well-studied notion of privacy, first proposed by Dwork, McSherry, Nissim, Smith (2006)
- Provides guarantees of how a *single* record influences the output of a mechanism
- Laplace mechanism: add noise to protect privacy

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Definition

A mechanism $M$ is $\epsilon$-differentially private if for databases $\mathcal{D}, \mathcal{D}'$ which differ in a single record, and for $r$ any output,

$$\frac{\Pr[M(\mathcal{D}) = r]}{\Pr[M(\mathcal{D}') = r]} \leq e^{\epsilon}$$
## Centralized vs. Distributed

- Usually, unprotected database located with a central party
- What if there is no trusted party?
- What algorithms can we give for the fully distributed setting?
Database Location

Centralized vs. Distributed

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- What algorithms can we give for the fully distributed setting?

Prior work

- Kasiviswanathan, Lee, Naor, et al. (2008) studied the fully distributed model in the context of learning
- McGregor, et al. (2008), studied the two database case
- Dwork, Naor, Pitassi, et al. (2009) studied heavy hitters in pan-private setting
## The Heavy Hitters problem

### Problem Statement
- Collection of users, each with a private universe element
- Goal: release the most popular element (the *heavy hitter*)
The Heavy Hitters problem

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Local Privacy Model
- No central authority has access to all the clean data
- Mechanism must query each user *individually* and return a universe element
- Each query must be differentially private
The Heavy Hitters problem

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**Questions:**
- What kind of accuracy is possible?
- Efficient algorithms?
Accuracy and Efficiency

\(\alpha\)-Accuracy

- If mechanism \(M\) returns an element whose frequency differs from the heavy hitter’s frequency by at most additive \(\alpha\), we say \(M\) is \(\alpha\)-accurate.
Accuracy and Efficiency

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Efficiency

- Notation: \(m\) number of users, \(N\) size of universe
- Consider \(N\) to be very large (number of websites on internet)
- Consider algorithm to be *efficient* if running time is \(\text{poly}(m, \log N)\)
Information theoretic results

**Theorem (Lower bound)**

There is no differentially private mechanism that achieves $\sqrt{m}$-accuracy for the heavy hitters problem with high probability, in the local model.

**Theorem (Upper bound)**

There is a differentially private algorithm that achieves $O(\sqrt{m \log N})$-accuracy for the heavy hitters problem with high probability, in the local model.
Theorem (Lower bound)

There is no differentially private mechanism that achieves $\sqrt{m}$-accuracy for the heavy hitters problem with high probability on the heavy hitters problem, in the local model.

Proof sketch

- Universe size $N = 2$, with users’ data drawn from a uniform distribution
- By differential privacy, belief about private data is approximately uniform given query answers
- By anti-concentration, mechanism can’t do better than $\sqrt{m}$ error with high probability
Lower bound on error

Comparison with centralized setting

- In centralized setting, can get $O(\log N)$-accuracy (exponential mechanism)
- $\Omega(\sqrt{m})$ error is *unavoidable* cost of moving to fully distributed setting
Near-optimal accuracy algorithm: JL-HH

Lemma (Johnson-Lindenstrauss)
For any set $S$ of $p$ points in $\mathbb{R}^w$, there is a linear map $A: \mathbb{R}^w \rightarrow \mathbb{R}^z$, where $z = O(\log(p)/\alpha^2)$, such that inner products are approximately preserved:

For any two points $u, v \in S$,

$$|\langle u, v \rangle - \langle Au, Av \rangle| \leq \alpha (\|u\|^2 + \|v\|^2)$$

Notation
Private histogram $v \in \mathbb{N}^N$, each $i$'th index contains count of element $i$.

Each user has histogram $u_i \in \mathbb{N}^N$, and $v = \sum_i u_i$.

Goal: return $\arg\max_i v_i$. 

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Near-optimal accuracy algorithm: JL-HH

### JL-HH sketch

- **Count** of $j$’th element is $\langle v, e_j \rangle$, with $e_j$ standard basis vector
- Estimate this by $\langle Av, Ae_j \rangle$
- Estimate $Av$ by summing $Au_i + \eta^i$ over all *users* $i$
- $\eta = \sum_i \eta^i$ noise to protect differential privacy
- For each universe element $j$, compute $\langle Av + \eta, Ae_j \rangle$
- Return element with largest estimated count

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**Accuracy, efficiency, and privacy**

Each user in JL-HH interacts in a differentially private way with the algorithm.

$O(\sqrt{m \log N})$-accurate for heavy hitters problem

Requires iterating over all $N$ universe elements, not efficient
### Near-optimal accuracy algorithm: JL-HH

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- Each user in JL-HH interacts in a differentially private way with the algorithm.
- $O(\sqrt{m \log N})$-accurate for heavy hitters problem
- Requires iterating over all $N$ universe elements, *not efficient*
Two incomparable, efficient algorithms

**Theorem (GLPS-HH Algorithm)**

There is a differentially private, efficient algorithm that achieves $O(m^{5/6})$-accuracy for the heavy hitters problem.

**Theorem (Bucket Algorithm)**

There is a differentially private, efficient algorithm that calculates the true heavy hitter with high probability, as long as the count of the heavy hitter dominates the $l_2$ norm of the other elements.
First efficient algorithm: GLPS-HH

**GLPS Algorithm**

- Gilbert, et al. (2009) give a sophisticated compressed sensing algorithm
- Similar idea as JL-HH: linear projection to lower dimensional space, add noise, then reconstruct the original histogram
- More technical decoding step to estimate histogram efficiently
- Runs in time $O(m \log^c N)$

**Theorem (Accuracy of GLPS-HH)**

GLPS-HH is $\alpha$-accurate for $\alpha = O\left(\frac{m^{5/6}}{\log^2 N}\right)$ with probability at least $\frac{3}{4}$. The failure probability can be driven down by iteration.
First efficient algorithm: GLPS-HH

GLPS Algorithm

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Second efficient algorithm: Bucket algorithm

Sketch of Bucket algorithm

- Take $\log N$ random hash functions, and hash each user’s data into one of two buckets.
- Total up noisy counts in each bucket, select the unique element that is hashed into the larger bucket by each hash function, if it exists.
- Run this procedure $\log N$ rounds, and take a majority vote to find the heavy hitter.
Bucket algorithm, in pictures

- Step 1: Select $\log N$ random 0/1 hash functions
Bucket algorithm, in pictures

- Step 1: Select $\log N$ random 0/1 hash functions
- Step 2: Hash user data into the buckets for each trial
Bucket algorithm, in pictures

- Step 1: Select \( \log N \) random 0/1 hash functions
- Step 2: Hash user data into the buckets for each trial
- Step 3: Total up noisy counts to find majority bucket
Bucket algorithm, in pictures

- Step 1: Select log $N$ random 0/1 hash functions
- Step 2: Hash user data into the buckets for each trial
- Step 3: Total up noisy counts to find majority bucket
- Step 4: Select element that hashes into majority bucket for each trial
Bucket algorithm: performance and runtime

Accuracy Guarantee: if heavy hitter has count greater than the $l_2$-norm of rest of histogram, algorithm will return true heavy hitter. No guarantee if condition is not met.

Privacy and running time

Bucket algorithm is differentially private. Pairwise independent hash functions suffice, linear hash functions finding element that hashes into all the larger buckets is fast: system of $O(\log N)$ linear equations.

Run time $O(m \log^3 N)$, efficient.

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- Bucket algorithm is differentially private
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Wrapping up

Open problems

- Are there algorithms that achieve optimal accuracy?
- What is the best that can be done efficiently (poly($m, \log N$) time)?
- What other problems in the distributed setting can be tackled with this approach?

Link to paper
